ISOLATED EXCEPTIONAL DEHN SURGERIES ON HYPERBOLIC KNOTS

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Dedicated to the memory of Yves Mathieu

Abstract. For a hyperbolic knot in the 3-sphere, at most finitely many Dehn surgeries yield non-hyperbolic manifolds. Such exceptional surgeries are classified into four types, lens space surgery, small Seifert fibered surgery, toroidal surgery and reducing surgery, according to the resulting manifolds. For each of the three types except reducing surgery, we give infinitely many hyperbolic knots with integral exceptional Dehn surgeries of the given type, whose adjacent integral surgeries are not exceptional.

1. Introduction

Let K be a knot in the 3-sphere S^3 with knot exterior $E(K) = S^3 - \operatorname{Int} N(K)$. A *slope* r on $\partial E(K)$ is an isotopy class of an essential simple closed curve on $\partial E(K)$. For a slope r, let $K(r)$ be the closed orientable 3-manifold obtained by *r-Dehn surgery* on K, that is, by attaching a solid torus V to $E(K)$ along their boundaries so that r bounds a disk in V. The slopes on $\partial E(K)$ are parameterized by the set $\mathbb{Q} \cup \{\infty\}$ in the usual way. By choosing a standard meridian-longitude basis $\{\mu, \lambda\}$ of $H_1(\partial E(K))$, a slope r corresponds to m/n if $[r] = m\mu + n\lambda$. The meridian slope ∞ is called a *trivial slope*. A slope is said to be *integral* if it corresponds to an integer. For two slopes r and s, their distance $\Delta(r, s)$ is the minimal geometric intersection number between them.

Suppose that K is hyperbolic, that is, the complement $S^3 - K$ admits a complete hyperbolic metric of finite volume. By Thurston's hyperbolic Dehn surgery theorem [23], all but finitely many surgeries on K yield hyperbolic 3-manifolds. We call a slope r *exceptional* if $K(r)$ is not hyperbolic. In this article, we will focus on integral exceptional slopes on hyperbolic knots. In fact, it is expected [14] that any non-trivial exceptional slope is integral, except Eudave-Muñoz knots [9], which are now known to be the only hyperbolic knots with non-integral toroidal surgeries [16].

Let us say that an exceptional slope r (and the corresponding surgery) is of *type* R, L, S, or T if $K(r)$ is reducible, a lens space, a Seifert fibered manifold over the 2-sphere with exactly three exceptional fibers, denoted by $S^2(p_1, p_2, p_3)$, or toroidal. Then it is known that if r is a non-trivial exceptional slope, then either r is of type R, L, S, T, or $K(r)$ gives a counterexample to the geometrization conjecture (see [14]). Furthermore, the famous cabling conjecture [13] claims that r does not happen to be of type R. Thus we can summarize that a non-trivial exceptional slope is expected to be of type L, S, or T.

An integral exceptional slope m is said to be *isolated* if both of $m-1$ and $m+1$ are not exceptional. In the literature, many examples of hyperbolic knots with exceptional slopes have appeared [3, 5, 9, 10, 11, 18]. As we will see in the next

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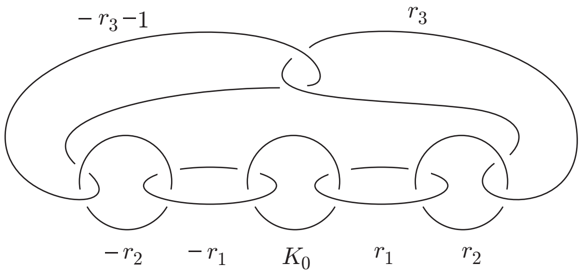

FIGURE 1. The surgery description for the knot K

section, hyperbolic knots with isolated exceptional slopes of type T are commonplace. However, as far as we know, there was no example with an isolated (integral) exceptional slope of type L or S. (We should remark that Eudave-Muñoz writes that John Dean has found knots with only one exceptional surgery, for example, the twisted torus knot $K(9,2,5,1)$ [11, page 121]. Eudave-Muñoz says that it was a private communication. In fact, 43-surgery on $K(9,2,5,1)$ can be confirmed to be of type S, and the computer program SnapPea written by Jeff Weeks suggests that the slope is isolated.)

The purpose of this article is to give the examples of hyperbolic knots with isolated exceptional slopes for each type of L, S, T.

Theorem 1.1. *For $X \in \{L, S, T\}$, there are infinitely many tunnel number one, hyperbolic knots in S^3 with an isolated integral exceptional slope of type X.*

2. PROOF OF THEOREM 1.1

In this section, we give a proof of Theorem 1.1. The proof is divided into three cases.

Case 2.1. $X = T$.

Let K be the 2-bridge knot corresponding to a continued fraction $[b_1, b_2]$, where b_1 and b_2 are even and $|b_1|, |b_2| > 2$. It is well known that K is of genus one, tunnel number one, and hyperbolic. Furthermore, slope 0 is the only non-trivial exceptional slope of K by [6]. In fact, it is of type T. Thus K has an isolated integral toroidal slope.

Case 2.2. $X = L$.

Let us consider the surgery description as shown in Figure 1, where r_i is a non-zero integer.

In addition to the indicated surgeries on the 6 components there, if we perform 0-surgery on K_0, then the resulting manifold is S^3. (It can be easily seen by Kirby calculus. See also [1].) Let K be the image of a meridian curve of K_0 after this surgery. Thus we can regard that the knot exterior $E(K)$ is obtained from the solid torus $S^3 - N(K_0)$ by performing those surgeries on the 6 components. Note that the slope 0 for K_0 corresponds to the trivial slope for K. Baker [1] shows that K belongs to a family of doubly primitive knots defined by Berge [2], which lie on a fiber surface of the left-handed trefoil. In particular, K has tunnel number one, because of doubly-primitiveness.

On the other hand, if we perform ∞-surgery on K_0 instead of 0-surgery, then the resulting manifold is a lens space, corresponding to the continued fraction

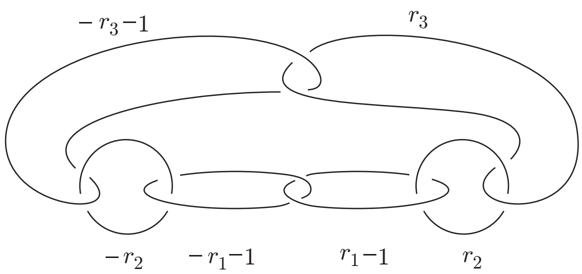

FIGURE 2. After 1-surgery on K_0

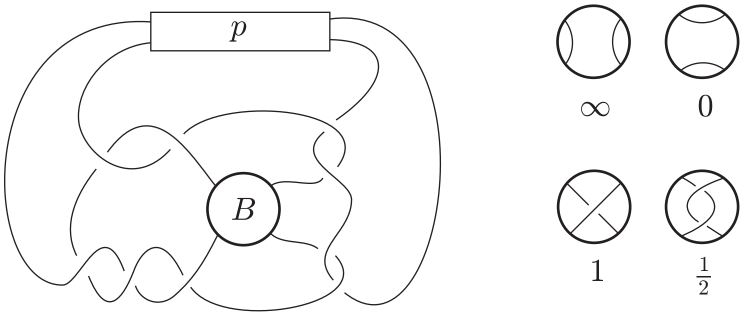

FIGURE 3. The tangle B_p

$[r_1, r_2, r_3, -r_3-1, -r_2, -r_1]$. Since the minimal geometric intersection number between the slopes 0 and ∞ is one, the slope ∞ for K_0 corresponds to some integral slope m for K. Hence the adjacent integral slopes to m with respect to K correspond to -1 and $+1$ for K_0.

If we perform 1-surgery on K_0, the surgery description can be changed to one as shown in Figure 2, by eliminating K_0. This 6-component link is the chain link $C(6,-4)$ of [21], which is shown to be hyperbolic there. Similarly, if we perform (-1)-surgery on K_0, then we obtain the chain link $C(6,-3)$, which is hyperbolic again. Thus we can choose $|r_1|, |r_2|, |r_3| \gg 0$ so that both 1-surgery and (-1)-surgery on K_0 yield hyperbolic manifolds by Thurston's hyperbolic Dehn surgery theorem.

Thus we have shown that K has an isolated integral slope of type L. This implies that K is hyperbolic as follows. For the torus knot of type (p,q), $pq \pm 1$ are the only integral slopes that yield lens spaces. However, slope pq yields a reducible manifold, and slopes $pq \pm 2$ yield Seifert fibered manifolds [20]. Hence K is not a torus knot. If K is a satellite knot with a lens space surgery, then K is a $(2, 2pq \pm 1)$-cable knot of a torus knot of type (p,q), and the lens space surgery corresponds to the slope $4pq \pm 1$ [4, 24]. Thus the lens space surgery is adjacent to the slope $4pq \pm 2$ which yields a reducible manifold.

Since a hyperbolic knot cannot admit two isolated exceptional surgeries of type L by the cyclic surgery theorem [7], infinitely many choices for r_i's give infinitely many hyperbolic knots.

Case 2.3. $X = S$.

Consider the tangle B_p in the 3-ball $S^3 - \operatorname{Int} B$ as illustrated in Figure 3, where the rectangle labeled by an integer p denotes p right-handed horizontal half twists.

Let $B_p(r)$ denote the knot or link obtained by inserting into the 3-ball B the rational tangle parameterized by $r \in \mathbb{Q} \cup \{\infty\}$. (We adopt the convention of [11] for

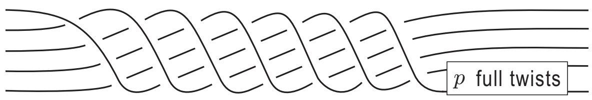

FIGURE 4. The knot K_p

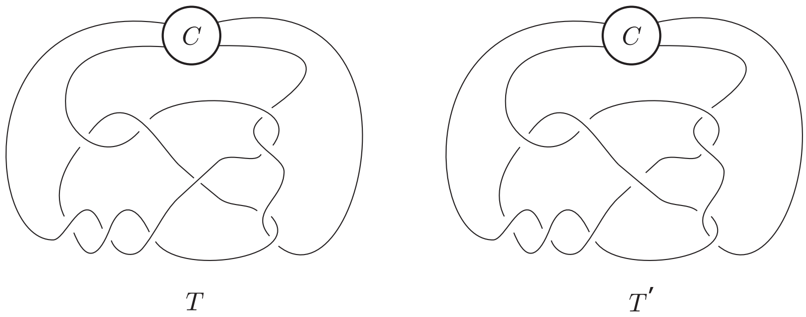

FIGURE 5. The tangles T and T'

the parameterization. See Figure 3.) Let M_p and $M_p(r)$ be the double branched coverings of B_p and $B_p(r)$, respectively.

Since $B_p(\infty)$ is the unknot, $M_p(\infty)$ is S^3, so M_p is the exterior of a knot, say K_p. We can see that $B_p(0)$ is the Montesinos link consisting of three rational tangles $(1/p, 2/5, -2/7)$. Hence $M_p(0)$ is the Seifert fibered manifold of type $S^2(p, 5, 7)$.

As in [9, 10], we can find an explicit description of the knot K_p. The knot K_p is obtained from the torus knot of type $(5, 7)$ by adding p-full twists on parallel two strings as shown in Figure 4. It is easy to see that K_p has tunnel number one by making use of an unknotting tunnel for the $(5, 7)$-torus knot. In fact, K_p is a twisted torus knot in the sense of Dean [8].

By keeping track of a latitude of B through the process of simplifying the unknot $B_p(\infty)$, we can see that the tangle slope 0 on B lifts to the slope $4p + 35$ on K_p with respect to its standard framing. Thus $K_p(4p + 35) = M_p(0)$.

To show that the slope $4p + 35$ is isolated for K_p, we need to show that $M_p(\pm 1)$ is hyperbolic. First, consider the tangle T in the 3-ball $S^3 - \operatorname{Int} C$ as illustrated in Figure 5. As before, $T(r)$ denotes the knot or link obtained by inserting into the 3-ball C the rational tangle parameterized by r. Note that $T(-p) = B_p(1)$. Let N and $N(r)$ be the double branched coverings of T and $T(r)$, respectively. Then $N(-p) = M_p(1)$.

Claim 2.4. The manifold N is hyperbolic.

Proof of Claim 2.4. We can see that $N(\infty)$ is the connected sum $L(2, 1)\sharp L(2, 1)$, $N(0)$ is the lens space $L(34, 9)$, $N(-1)$ is a Seifert fibered manifold of type $S^2(2, 3, 4)$, and $N(-1/2)$ is a non-Seifert fibered, irreducible toroidal manifold as shown in Figure 6.

Since $N(-1)$ and $N(0)$ are non-homeomorphic prime manifolds, N is irreducible. If N is boundary-reducible, then N would be a solid torus. This is impossible, because $N(\infty)$ is not a lens space. If N is Seifert fibered, then $N(r)$ is Seifert fibered for all but at most one r, for which $N(r)$ is reducible. Since $N(-1/2)$ is irreducible and not a Seifert fibered manifold, this is impossible. Thus it remains to prove that N is atoroidal.

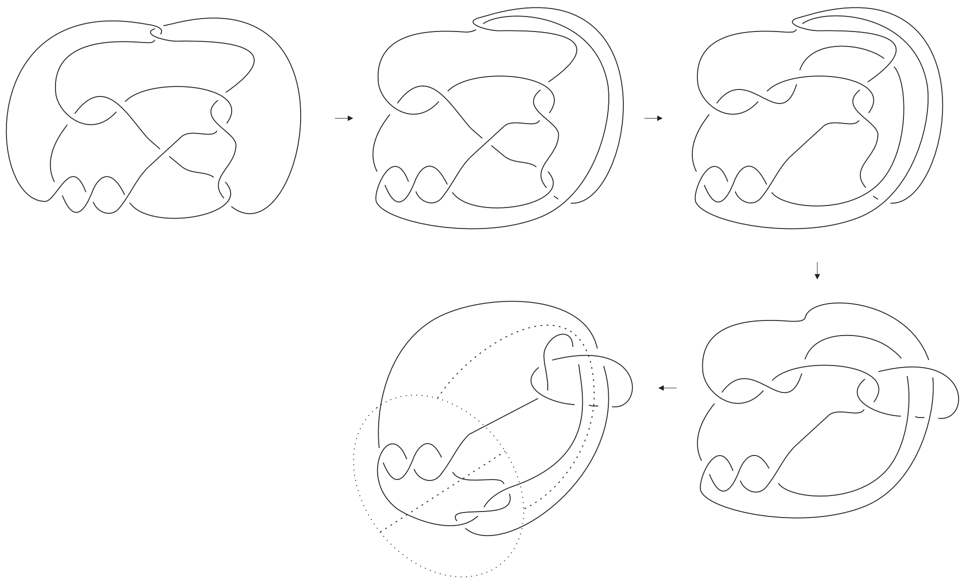

FIGURE 6. $N(-1/2)$

We will follow the argument of the proof of [12, Theorem 4.2]. Assume that N contains an essential torus S. Since $N(0)$ is a lens space, S is separating. Let W be the part between S and ∂N. Also, S is compressible in $N(\infty)$, $N(-1/2)$ and $N(0)$. Thus W is a cable space $C(r,s)$ with cabling slope r_0 on ∂N, since $\Delta(\infty,-1/2)=2$ [12, Lemma 2.4]. Solving the equation $\Delta(r_0,\infty)=\Delta(r_0,-1/2)=1$, $r_0=0$ or -1. However, $r_0=0$, because $N(-1)$ does not contain a lens space summand.

Let δ_0 and δ_∞ be the slopes on S which bound disks in $W(0)$ and $W(\infty)$, respectively. Since 0 is the cabling slope, $N(0)=L(r,s)\sharp W'(\delta_0)$, where $W'=N-\operatorname{Int}W$, and $W'(\delta_0)$ denotes δ_0-Dehn filling on W' along S. Since $N(0)$ is a lens space, W' is the knot exterior of a knot in S^3 with meridional slope δ_0. Then $N(\infty)=W'(\delta_\infty)=L(2,1)\sharp L(2,1)$. This is impossible, because $L(2,1)\sharp L(2,1)$ has non-cyclic 1-dimensional homology group. $\square$

Thus $N(-p)=M_p(1)$ is hyperbolic except for finitely many integers p by the hyperbolic Dehn surgery theorem.

Similarly, consider the tangle T' as shown in Figure 5. Let N' be the double branched covering of T'. Then $N'(\infty)$ is a Seifert fibered manifold of type $S^2(2,2,2)$, $N'(0)=L(34,9)$, $N'(1)=L(32,9)$, and $N'(1/2)$ is a non-Seifert fibered, irreducible, toroidal manifold. As above, N' can be seen to be irreducible, boundary-irreducible, and non-Seifert fibered. To prove that N' is atoroidal, suppose that N' contains an essential torus. Then N' is decomposed into the union of a cable space $W=C(r,s)$ and W' again. Here, we may suppose that W' is atoroidal by choosing an "outermost" essential torus in N' with respect to the torus decomposition of N' (see [17, Lemma 23.3(2)]). There are two possibilities for the cabling slope r_0, 0 and 1, as the solutions of $\Delta(r_0,\infty)=\Delta(r_0,1/2)=1$. In either case, let δ_0 be the slope which bounds a disk in $W(r_0)$. Then $N'(r_0)=L(r,s)\sharp W'(\delta_0)$. Since $N'(0)$ and $N'(1)$ are lens spaces, W' is the knot exterior of a knot in S^3 with meridional slope δ_0.

On the other hand, $W(1/2)$ is a solid torus, since $\Delta(r_0,1/2)=1$. Let $\delta_{1/2}$ be the slope on $\partial W(1/2)$ which bounds a disk in $W(1/2)$. Thus $N'(1/2)=W'(\delta_{1/2})$.

Since $N'(1/2)$ is toroidal and $\Delta(\delta_0, \delta_{1/2}) = r > 2$, W' must be the knot exterior of a satellite knot [15]. This contradicts that W' is atoroidal. Hence we have shown that N' is hyperbolic. Then $N'(-p) = M_p(-1)$ is hyperbolic except for finitely many integers p by the hyperbolic Dehn surgery theorem again.

Thus we have shown that both $M_p(1)$ and $M_p(-1)$ are hyperbolic except for finitely many integers p. Finally, we confirm that the knot K_p is hyperbolic under such a choice of p.

Claim 2.5. For any integer p such that $M_p(1)$ and $M_p(-1)$ are hyperbolic, K_p is hyperbolic.

Proof of Claim 2.5. If K_p is a torus knot, then $K_p(4p + 35 \pm 1)$ is not hyperbolic [20]. Hence K_p is not a torus knot.

Next, suppose that K_p is a satellite knot. Since K_p has tunnel number one, it has a torus knot as a companion [19]. Let T be the essential torus in $E(K_p)$ which bounds the torus knot exterior. In $K_p(4p + 35 \pm 1)$, T is compressible. By [22], T bounds either a solid torus or the connected sum of a solid torus and a lens space in $K_p(4p + 35 \pm 1)$. In the former, $K_p(4p + 35 \pm 1)$ is obtained from Dehn surgery on a torus knot. However, no Dehn surgery on a torus knot yields a hyperbolic manifold. The latter is also impossible, because $K_p(4p + 35 \pm 1)$ is irreducible, and not a lens space. $\square$

This completes the proof of Theorem 1.1 for the case of type S.

We remark that SnapPea suggests that K_p is hyperbolic whenever $p \neq 0$.

3. Remark and question

Berge [2] introduced the notion of doubly primitive knots, and described twelve families of doubly primitive knots, called Berge knots. It is conjectured that Berge knots comprise all knots admitting lens space surgeries. Except the two families of Berge knots, referred to as families (VII) and (VIII) as in [1], which lie on the fiber surface of the trefoil or the figure-eight knot, we can verify that no lens space surgery is isolated. Baker [1] shows that any knot in families (VII) and (VIII) has a surgery description on a minimally twisted chain link with an odd number of components as in Figure 1. Generically, it seems that a knot in those families admits an isolated lens space surgery, if the corresponding chain link has at least seven components.

As far as we know, when a hyperbolic knot admits multiple integral exceptional slopes, these slopes are successive.

Question 3.1. Are integral exceptional slopes for a hyperbolic knot successive?

We expect that any knot given in Section 2 has exactly one non-trivial exceptional slope.

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