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Failure-Correlated Opportunity-based Age Replacement Models

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In this paper we extend the existing opportunity-based age replacement policies by taking account of dependency between the failure time and the arrival time of a replacement opportunity for one unit system. Based on the bivariate probability distribution function of the failure time and the arrival time of the opportunity, we focus on two opportunity-based age replacement problems and characterize the cost-optimal age replacement policies which minimize the relevant expected costs, with the hazard gradient, which is a vector-valued bivariate hazard rate. Through numerical examples with the Farlie-Gumbel-Morgenstern bivariate copula and the Gaussian bivariate copula having the general marginal distributions, we investigate the dependence of correlation between the failure time and the opportunistic replacement time on the age replacement policies.

Keywords: opportunity-based age replacement; correlation; bivariate distribution; copula; hazard gradient; initial hazard rate function; expected cost per unit time.

1. Introduction

Age replacement[12] is the simplest but most realistic replacement policy to real applications, because the pre-scheduled preventive replacement of unfailed units or components is often feasible and beneficial from the viewpoint of maintenance management cost. Nakagawa[11] calls the replacement policies which are triggered at random timing the *random replacement policies*. Traditionally, such replacement models have been referred to as *opportunity-based replacement models* in the sense that an opportunity to trigger the preventive replacement arrives at random times. Pullen and Thomas[14] consider an opportunistic replacement policy for a two-unit system. Zheng and Fard[20,21] and Zheng[22] also apply opportunity-based replacement policy to the other maintenance models. Dekker and Smeitink[3,5] and Dekker and Dijkstra [4] give the mathematical bases for optimization of both age-type and block-type of opportunity-based replacement policies. Iskandar and Sando [7] and Dohi et al.[6] consider somewhat different opportunity-based age replacement policies from Dekker and Dijkstra[4] in continuous and discrete time setting, respectively. Recently, Okamura and Dohi[13] apply an idea of opportunity-based age replacement model[4] to determine the optimal timing when to rejuvenate operating software systems with degradation, called the software aging.

On the other hand, Nakagawa[11] summarizes many model variations which contain both of opportunity-based replacement and pre-scheduled replacement policies such as age replacement, block replacement, periodic replacement. More specifically, Nakagawa and Zhao[10], Zhao and Nakagawa[17] and Zhao et al.[18] introduce the concept of *replacement first* and *replacement last*, and compare these two replacement schemes in different model setting. Zhao et al.[19] also analyze the similar opportunity-based replacement policies in a different context. In all the references listed above, it is assumed that the failure time is statistically independent of the arrival time of replacement opportunity. However, it should be noted that this is a rather strong assumption in reality. For instance, consider a fully automated manufacturing system, where one of two preventive replacement options is possible; annual replacement and opportunity-based replacement of a unit in the system. Under the assumption that the opportunity-based age replacement cost is much cheaper than the preventive age replacement cost, if the system operator allows to trigger the opportunity-based age replacement at an unknown timing, he or she may carefully monitor the unit age, and expect an earlier arrival of the opportynity than the pre-scheduled replacemen time. Also, when the failure is caused by human errors, which may occur in the arrangement of replacement, the failure time must be strongly correlated with the arrival of replacement opportunity. To our best knowledge, no work on the failure-correlated opportunity-based replacement model has been reported yet in the literature. In order to describe the correlation between the failure time and the arrival time of replacement opportunity, we introduce the bivariate copula with arbitrary marginal distributions, and consider the failure-correlated opportunity-based age replacement policies. In this paper, we just focus on two opportunity-based replacement models in Zhao and Nakagawa[17]; opportunity-based age replacement first and opportunity-based age replacement last policies, and investigate the dependence of correlation on the replacement decision.

The paper is organized as follows. In Sections 2 and 3, we consider the opportunity-based age replacement first and the opportunity-based age replacement last, where the resulting opportunity-based age replacement models are extensions of the existing models[10,17] by taking account of dependency between the failure time and the arrival time of a replacement opportunity for one unit system. Based on an arbitrary bivariate probability distribution function of the failure time and the arrival time of the opportunity, we characterize the cost-optimal age replacement first policy which minimizes the relevant expected cost, with the hazard gradient[9], which is a vector-valued bivariate hazard rate[8] . It is confirmed that the analytical result for the opportunity-based age replacement last is not applicable under the plausible condition. Through numerical examples with the Farlie-Gumbel-Morgenstern (FGM) bivariate copula and the Gaussian bivariate copula, we investigate the dependence of correlation between the failure time and the opportunistic replacement time on the two opportunity-based age replacement policies.

2. Opportunity-based Age Replacement First

2.1. *Formulation*

Let X be an absolutely continuous non-negative random variable to denote the failure time of one unit, where $\Pr\{X \leq x\} = F(x)$, $f(x) = dF(x)/dx$ and $\mu = \int_0^\infty x dF(x)$ are the cumulative distribution function (c.d.f.), probability density function (p.d.f.) and mean time-to-failure (MTTF), respectively. Suppose that there are three possible options on replacement. If the unit fails at time X, then the failed unit is replaced by a new one with the failure replacement cost c_F (> 0). We also suppose that the non-failed unit is replaced at a pre-scheduled preventive replacement time t_0 (> 0) with preventive replacement cost c_T (> 0) or a random opportunistic replacement time Y with opportunity-based replacement cost c_R (> 0), where Y is also an absolutely continuous non-negative random variable having c.d.f. $\Pr\{Y \leq y\} = G(y)$, p.d.f. $g(y) = dG(y)/dy$ and mean time-to-opportunistic replacement (MTTOR), $r = \int_0^\infty y dG(y)$. Hence the replacement is made at the failure time (failure replacement) or one of two preventive replacement times, whichever occurs first, i.e., at time $\min\{X, \min(t_0, Y)\} = \min\{X, Y, t_0\}$. Hereafter we call this age replacement policy the *opportunity-based age replacement first*[11]. To motivate the opportunistic replacement arrived at random timing Y, it is assumed without any loss of generality that $c_F > c_T > c_R > 0$.

Suppose that the random variable X statistically depends on Y. Define the joint distribution:

$$\Pr\{X \leq x, Y \leq y\} = P(x,y) = \int_0^x \int_0^y p_{X,Y}(s,t) ds dt, \tag{1}$$

where $p_{X,Y}(x,y) = \partial^2 P(x,y)/\partial x \partial y$ is the bivariate p.d.f of (X,Y), $\lim_{y\to\infty} P(x,y) = F(x)$ and $\lim_{x\to\infty} P(x,y) = G(y)$ are the marginal c.d.f.'s. It is well-known that the bivariate survivor function is given by

$$\Pr\{X > x, Y > y\} = S(x,y) = 1 - F(x) - G(y) + P(x,y). \tag{2}$$

If the bivariate c.d.f in Eq. (1) is absolutely continuous, the long-run average cost in the steady state for our opportunity-based age replacement first is given by

$$\begin{aligned} C_1(t_0) &= \lim_{t\to\infty} \frac{\mathrm{E}\big[\text{total cost on } [0,t)\big]}{t} \\ &= \frac{c_F \Pr\{X \leq t_0, X \leq Y\} + c_R \Pr\{Y \leq t_0, Y \leq X\}}{\mathrm{E}[\min\{X, Y, t_0\}]} \\ &\quad + \frac{c_T[1 - \Pr\{X \leq t_0, X \leq Y\} - \Pr\{Y \leq t_0, Y \leq X\}]}{\mathrm{E}[\min\{X, Y, t_0\}]} \\ &= \frac{c_T + (c_F - c_T)\int_0^{t_0}\int_x^\infty p_{X,Y}(x,y) dy dx}{\int_0^{t_0} S(t,t) dt} \\ &\quad - \frac{(c_T - c_R)\int_0^{t_0}\int_y^\infty p_{X,Y}(x,y) dx dy}{\int_0^{t_0} S(t,t) dt}. \end{aligned} \tag{3}$$

Define the numerator and denominator of Eq. (3) by $A_1(t_0)$ and $B_1(t_0)$, respectively, where $A_1(0) = c_T$,

$$A_1(\infty) = c_T + (c_F - c_T)\int_0^\infty \int_x^\infty p_{X,Y}(x,y)dydx \\ -(c_T - c_R)\int_0^\infty \int_y^\infty p_{X,Y}(x,y)dxdy, \quad (4)$$

$B_1(0) = 0$ and $B_1(\infty) = \int_0^\infty S(t,t)dt$. Differentiating $C_1(t_0)$ with respect to t_0 and dividing it by $B_1^2(t_0)$ yield $dC_1(t_0)/dt_0 = q_1(t_0)/B_1^2(t_0)$, where

$$q_1(t_0) = \Big\{(c_F - c_T)\lambda_X(t_0) - (c_T - c_R)\lambda_Y(t_0)\Big\}B_1(t_0) - A_1(t_0). \quad (5)$$

In the independent case[10,17], it is common to characterize the optimality equation with the hazard rate functions, $f(t_0)/\overline{F}(t_0)$ and $g(t_0)/\overline{G}(t_0)$, where in general $\overline{\psi(\cdot)} = 1-\psi(\cdot)$. In the bivariate case, the bivariate hazard rate in the sense of Basu[1] is well-known, but does not work to characterize our optimality equation in the dependent case. In Eq. (5), the functions

$$\lambda_X(t) = \int_t^\infty p_{X,Y}(t,y)dy/S(t,t), \quad \lambda_Y(t) = \int_t^\infty p_{X,Y}(x,t)dx/S(t,t) \quad (6)$$

are called the *initial hazard rate functions*[15] for the bivariate random variable (X,Y). Johnson and Kotz[8] and Marshall[9] define the hazard gradient which denotes a vector multivariate hazard rate. Shaked and Shanthikumar[16] and later Scarsini and Shaked[15] define the initial hazard rate functions for the hazard gradient, and show that they play a significant role to characterize the multivariate lifetime.

We are in the position to characterize the optimal opportunity-based age replacement first with the initial hazard rate functions:

Theorem 1. *(1) Suppose that $(c_F - c_T)\lambda'_X(t_0) > (c_T - c_R)\lambda'_Y(t_0)$, where $\lambda'_X(t) = d\lambda_X(t)/dt$ and $\lambda'_Y(t) = d\lambda_Y(t)/dt$. If $q_1(\infty) > 0$, then there exists a unique optimal age replacement time t_0^* $(0 < t_0^* < \infty)$ minimizing $C_1(t_0)$, where*

$$C_1(t_0^*) = (c_F - c_T)\lambda_X(t_0^*) - (c_T - c_R)\lambda_Y(t_0^*). \quad (7)$$

Otherwise, i.e., $q_1(\infty) \le 0$, the function $C_1(t_0)$ decreases in t_0 and the optimal age replacement time is given by $t_0^* \to \infty$ with $C_1(\infty) = A_1(\infty)/B_1(\infty)$.
(2) Suppose that $(c_F - c_T)\lambda'_X(t_0) \le (c_T - c_R)\lambda'_Y(t_0)$. Then the optimal age replacement time is given by $t_0^* \to \infty$.

Proof. Taking the first derivative of $q_1(t_0)$ with respect to t_0 implies

$$\frac{dq_1(t_0)}{dt_0} = \Big\{(c_F - c_T)\lambda'_X(t_0) + (c_R - c_T)\lambda'_Y(t_0)\Big\}B_1(t_0). \quad (8)$$

Under the assumption of $c_F > c_T$ and $c_R > c_T$, if $(c_F - c_T)\lambda'_X(t_0) > (c_T - c_R)\lambda'_Y(t_0)$, then $dq_1(t_0)/dt_0 > 0$, otherwise $dq_1(t_0)/dt_0 \le 0$. From $q_1(0) = -c_T < 0$, if $q_1(\infty) > 0$, then there exists a unique optimal age replacement time t_0^* $(0 <$

$t_0^* < \infty$) satisfying $q_1(t_0) = 0$, otherwise, the function $q_1(t_0)$ is always negative and $C_1(t_0)$ is a decreasing function of t_0. Hence the proof is completed. □

Corollary 1. *In the simplest case where X and Y are statistically independent, say, $P(x, y) = F(x)G(y)$, the long-run average cost is given by*

$$C_1(t_0) = \frac{c_T + (c_F - c_T)\int_0^{t_0} \overline{G}(t)dF(t) - (c_T - c_R)\{1 - \overline{F}(t_0)\overline{G}(t_0)\}}{\int_0^{t_0} \overline{G}(t)\overline{F}(t)dt}. \tag{9}$$

The result is trivial and can be seen in Nakagawa[11]*, because*

$$\lambda_X(t_0) = f(t_0)/\overline{F}(t_0), \tag{10}$$

$$\lambda_Y(t_0) = g(t_0)/\overline{G}(t_0). \tag{11}$$

Next we show that the opportunity-based age replacement first[11] is equivalent to a simple two-unit system's replacement (see *e.g.* Chopra and Ram[2]). Consider a two-unit series system which consists of Unit 1 and Unit 2 with the bivariate failure time (X, Y), respectively. Let c_i $(i = 1, 2)$ denote the replacement cost for Unit i if it fails. In this case, another operative Unit $3 - i$ is also replaced at the failure time for an opposite Unit i from the view point of preventive maintenance. The preventive replacement of two operative units is made at time t_0 with cost c_T even though they do not fail. Then, setting $c_F = c_1$ and $c_R = c_2$, we have

$$\begin{aligned} C_1(t_0) = & \frac{c_T + (c_1 - c_T)\int_0^{t_0}\int_x^{\infty} p_{X,Y}(x, y)dxdy}{\int_0^{t_0} S(t, t)dt} \\ & - \frac{(c_T - c_2)\int_0^{t_0}\int_y^{\infty} p_{X,Y}(x, y)dxdy}{\int_0^{t_0} S(t, t)dt}, \end{aligned} \tag{12}$$

$$q_1(t_0) = \Big\{(c_1 - c_T)\lambda_X(t_0) + (c_2 - c_T)\lambda_Y(t_0)\Big\}B_1(t_0) - A_1(t_0). \tag{13}$$

Hence, if $c_i > c_T$ $(i = 1, 2)$ with $\lambda'_X(t) > 0$ and $\lambda'_Y(t) > 0$, then $dq_1(t_0)/dt_0 > 0$, so that our extended opportunity-based age replacement first reduces to a simple age replacement for a correlated two-unit system under a milder assumption.

3. Opportunity-based Age Replacement Last

Noting the cost assumption on $c_F > c_T > c_R$ in the opportunity-based age replacement first, it may not be always better to trigger the preventive replacement at time t_0, provided that both of failure and arrival of opportunistic replacement do not occur when c_R is relatively smaller than c_T. Zhao and Nakagawa[17] consider a somewhat different opportunity-based age replacement policy which is called the *opportunity-based age replacement last.* In this type of opportunity-based age replacement, the preventive replacement is made at time t_0 or Y whichever occurs last. Under the same cost assumption as Section 2, we extend the result in Zhao and Nakagawa[17] by taking account of the dependency of (X, Y) under $c_T \neq c_R$.

The expected cost per unit time in the steady state for our opportunity-based age replacement last is given by

$$\begin{aligned} C_2(t_0) &= \frac{c_R \Pr\{Y \geq t_0, Y \leq X\} + c_T \Pr\{Y \leq t_0, t_0 \leq X\}}{\mathrm{E}[\min\{X, \max(Y, t_0)\}]} \\ &\quad + \frac{c_F[1 - \Pr\{Y \geq t_0, Y \leq X\} - \Pr\{Y \leq t_0, t_0 \leq X\}]}{\mathrm{E}[\min\{X, \max(Y, t_0)\}]} \\ &= \frac{c_F - (c_F - c_T)\int_0^{t_0}\int_{t_0}^{\infty} p_{X,Y}(x,y)dxydy}{\int_0^{t_0}\overline{F}(t)dt + \int_{t_0}^{\infty} S(t,t)dt} \\ &\quad - \frac{(c_T - c_R)\int_{t_0}^{\infty}\int_y^{\infty} p_{X,Y}(x,y)dxdy}{\int_0^{t_0}\overline{F}(t)dt + \int_{t_0}^{\infty} S(t,t)dt}. \end{aligned} \tag{14}$$

If X is independent of Y, then we have

$$C_2(t_0) = \frac{c_F - (c_F - c_T)\overline{F}(t_0)G(t_0) - (c_F - c_R)\int_{t_0}^{\infty}\overline{F}(y)dG(y)}{\int_0^{t_0}\overline{F}(t)dt + \int_{t_0}^{\infty}\overline{F}(t)\overline{G}(t)dt}, \tag{15}$$

which can be reduced to Eq. (15) in Zhao and Nakagawa[17] when $c_T = c_R$. Unfortunately, it seems impossible to show analytically the uniqueness of the optimal opportunity-based age replacement last policy even in the independent case, in spite that the cost assumption $c_T > c_R$ is essential. But it is quite easy to show the existense of a unique and finite solution numerically.

4. Numerical Examples

4.1. *Preliminary*

In the numerical examples, we consider the bivariate copula to represent the correlation between failure time and opportunity. The copula is a multivariate probability distribution when its marginal distributions follow uniform distributions, which can describe the dependence between variables without parametric forms of marginal distributions. For the bivariate distribution, the copula is defined by a function $C(u, v)$, $0 \leq u \leq 1$, $0 \leq v \leq 1$, which maps to the real interval $[0, 1]$. Concretely, the joint distribution can be represented in the form:

$$\Pr\{X \leq x, Y \leq y\} = C(F_X(x), F_Y(y)), \tag{16}$$

where $F_X(x) = \Pr\{X \leq x\}$ and $F_Y(y) = \{Y \leq y\}$ are the c.d.f.'s of the marginal distributions. On the other hand, the survival copula, $\overline{C}(u, v)$, is the copula to represent the dependence of variables in the joint survival function. From the argument of joint distribution, the relationship between the original copula and the survival copula is given by

$$\overline{C}(u, v) = u + v - 1 + C(1 - u, 1 - v), \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1. \tag{17}$$

By using the survival copula, the joint survival function can be written as

$$\Pr\{X > x, Y > y\} = \overline{C}(\overline{F}_X(x), \overline{F}_Y(y)). \tag{18}$$

4.2. *Farlie-Gumbel-Morgenstern copula*

In the first example, we consider the Farlie-Gumbel-Morgenstern (FGM) bivariate copula to represent the dependence between the failure time and the arrival time of an opportunity. The FGM bivariate copula is given by

$$C(u,v) = uv(1+\alpha(1-u)(1-v)), \quad 0 \le u \le 1, \quad 0 \le v \le 1, \tag{19}$$

where $-1 \le \alpha \le 1$ is a parameter for the strength of correlation. The survival copula of the FGM copula is the same as the original copula. Spearman's rank correlation coefficient becomes $\rho_S = \alpha/3$, i.e, Spearman's rank correlation coefficient is always in the range $-1/3 \le \rho_S \le 1/3$. Also, the lower and upper tail dependence coefficients become 0. It is well known that the FGM copula cannot represent the strength correlation compared to other copulas.

According to the FGM bivariate copula, we suppose the joint survival function of failure time and opportunity arrival time (X, Y) as follows.

$$S(x,y) = \overline{F}(x)\overline{G}(y)\{1 + \alpha F(x)G(y)\}. \tag{20}$$

In our numerical examples, we assume that the marginal c.d.f. of X is given by the Weibull distribution with mean $\mu = 10$ and shape parameter 2. Also the marginal c.d.f. of Y is assumed to be the exponential distribution with mean $r = 1, 5, 10$. The cost parameters are set as $c_R = 1$, $c_T = 2, 5$ and $c_F = 10$.

Tables 1 and 2 present the optimal opportunity-based age replacement time t_0^* and the minimum expected cost under the opportunity-based age replacement first and last policies with $c_T = 2$ and $c_T = 5$. In the tables, the column ρ_S indicates the corresponding Spearman's rank correlation coefficients of X and Y.

In both tables, it can be seen that, when the correlation is positive, the minimum expected cost becomes smaller in the both opportunity-based age replacement policies. This is a plausible observation because the replacement opportunity with positive correlation tends to occur in accordance with the unit aging. The negative correlation implies the opposite case. In this case, when the unit does not fail for long time period, the replacement opportunity may arrive earlier. Since the unit should be replaced at the opportunity arrival under the replacement first policy, the cost gets worse than the ordinary age replacement. Also, under the replacement last policy, since the arrival of opportunity is delayed as the failure time becomes smaller, the replacement time should be shorter. This is because the policy gets close to the opportunistic replacement policy without the scheduled replacement. However, since the difference between the failure time and the arrival time of opportunity is large in the case of negative correlation, the opportunistic replacement policy does not work well. In other words, when the correlation is negative, there are the cases where the age replacement first and last do not work even if the cost of opportunistic replacement is smaller than the preventive age replacement cost. In fact, when the ordinary age replacement is applied, the optimal replacement time t_0^* and the minimum expected cost $C(t_0^*)$ are $t_0^* = 5.7621$ and $C(t_0^*) = 0.7241$ for $c_T = 2$, and $t_0^* = 12.3083$ and $C(t_0^*) = 0.9667$ for $c_T = 5$. In Table 1, there is no case where the opportunity-based age replacement is superior to the ordinary age replacement if the correlation is negative. On the other hand, the optimal opportunity-based age replacement time under the replacement first is not monotone in terms of the correlation coefficient. In Table 2, the optimal opportunity-based age replacement time decreases in the case where the correlation coefficient becomes larger.

The superiority of the opportunity-based age replacement first (last) in the sense of minimization of the long-run average cost depends on model parameters; MTTOR and the correlation. It should be noted that the opportunity-based age replacement last has to skip the first option of preventive replacement; pre-scheduled preventive replacement or opportunistic preventive replacement, so the probability that the failure occurs after this skip is positive. In our examples, it is found that the case where the opportunity-based age replacement last works better corresponds to the case where the opportunity-based age replacement first policy is equivalent to the failure replacement. Hence, when the correlation is positive, if the opportunity-based age replacement first policy becomes the failure replacement, say, $t_0^* \to \infty$, then it may be feasible to trigger the opportunity-based age replacement last policy, otherwise, the opportunity-based age replacement first is reasonable. On the other hand, as c_T increases, i.e., as the preventive replacement cost is relatively higher than the opportunity-based replacement cost c_R, the opportunistic preventive replacement is preferred. In our example, the difference between the minimum costs under replacement first and the replacement last becomes small in the case of $c_T = 5$.

4.3. *Gaussian copula*

Next we consider the Gaussian bivariate copula to represent the dependence between failure time and opportunity. The Gaussian bivariate copula is given in the following form:

$$C(u, v) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v); \rho), \quad 0 \le u \le 1, \quad 0 \le v \le 1, \tag{21}$$

where $\Phi_2(\cdot, \cdot; \rho)$ is the joint c.d.f. of standard bivariate normal distribution with the (Pearson) correlation parameter ρ, and $\Phi(\cdot)$ is the c.d.f. of the standard (univariate) normal distribution. Spearman's rank correlation coefficient of the Gaussian copula is given by $\rho_S = \frac{6}{\pi} \arcsin(\rho/2)$, and thus the possible range is $-1 \le \rho_S \le 1$. Thus the Gaussian copula can represent the strong dependence between random variables. On the other hand, the lower and upper dependence coefficients are always 0.

From the Gaussian bivariate copula, the joint survival function of failure time and arrival time of an opportunity (X, Y) becomes

$$S(x, y) = \overline{F}(x) + \overline{G}(y) - 1 + \Phi_2(\Phi^{-1}(F(x)), \Phi^{-1}(G(y)); \rho). \tag{22}$$

Similar to the case of FGM bivariate copula, we assume that the marginal c.d.f. of X is given by the Weibull distribution with mean $\mu = 10$ and shape parameter 2 and that the marginal c.d.f. of Y is the exponential distribution with mean $r = 1, 5, 10$. The cost parameters are also set as $c_R = 1$, $c_T = 2, 5$ and $c_F = 10$.

Table 1. Optimal opportunity-based age replacement time with FGM copula ($c_R = 1$, $c_T = 2$, $c_F = 10$).

		Replacement First		Replacement Last	
$MTTOR$	ρ_S	t_0^*	$C_1(t_0^*)$	t_0^*	$C_2(t_0^*)$
1	0.3	∞	1.0585	5.7116	0.7234
1	0.2	∞	1.0896	5.7189	0.7235
1	0.1	∞	1.1209	5.7262	0.7236
1	0.0	∞	1.1525	5.7333	0.7237
1	-0.1	∞	1.1843	5.7404	0.7238
1	-0.2	∞	1.2163	5.7473	0.7239
1	-0.3	∞	1.2485	5.7542	0.7240
5	0.3	8.6585	0.5623	0.0000	0.5982
5	0.2	8.3043	0.6062	0.0000	0.6406
5	0.1	7.7852	0.6507	0.0000	0.6853
5	0.0	7.1189	0.6946	0.0000	0.7323
5	-0.1	6.4212	0.7367	5.6951	0.7535
5	-0.2	5.8117	0.7762	6.2739	0.7648
5	-0.3	5.3229	0.8126	6.7322	0.7736
10	0.3	7.5509	0.6151	0.0000	0.7264
10	0.2	7.1923	0.6447	0.0000	0.7532
10	0.1	6.7891	0.6735	0.0000	0.7816
10	0.0	6.3764	0.7013	5.2860	0.8078
10	-0.1	5.9880	0.7277	6.3593	0.8189
10	-0.2	5.6423	0.7526	7.1613	0.8259
10	-0.3	5.3429	0.7760	7.7661	0.8300

Table 2. Optimal opportunity-based age replacement time with FGM copula ($c_R = 1$, $c_T = 5$, $c_F = 10$).

		Replacement First		Replacement Last	
$MTTOR$	ρ_S	t_0^*	$C_1(t_0^*)$	t_0^*	$C_2(t_0^*)$
1	0.3	∞	1.0585	12.3079	0.9667
1	0.2	∞	1.0896	12.3080	0.9667
1	0.1	∞	1.1209	12.3080	0.9667
1	0.0	∞	1.1525	12.3081	0.9667
1	-0.1	∞	1.1843	12.3081	0.9667
1	-0.2	∞	1.2163	12.3082	0.9667
1	-0.3	∞	1.2485	12.3083	0.9667
5	0.3	18.3139	0.5979	0.0000	0.5982
5	0.2	18.7061	0.6405	0.0000	0.6406
5	0.1	19.1100	0.6852	0.0000	0.6853
5	0.0	19.5092	0.7323	0.0000	0.7323
5	-0.1	19.8493	0.7819	0.0000	0.7819
5	-0.2	19.8509	0.8342	0.0000	0.8342
5	-0.3	15.3205	0.8895	0.0000	0.8896
10	0.3	14.8783	0.7202	0.0000	0.7264
10	0.2	15.0660	0.7482	0.0000	0.7532
10	0.1	15.2438	0.7777	0.0000	0.7816
10	0.0	15.3918	0.8089	0.0000	0.8117
10	-0.1	15.4578	0.8418	0.0000	0.8438
10	-0.2	15.2812	0.8768	0.0000	0.8781
10	-0.3	14.3305	0.9137	0.0000	0.9147

Table 3. Optimal opportunity-based age replacement time with Gaussian copula ($c_R = 1$, $c_T = 2$, $c_F = 10$).

		Replacement First		Replacement Last	
$MTTOR$	ρ_S	t_0^*	$C_1(t_0^*)$	t_0^*	$C_2(t_0^*)$
1	0.3	∞	1.0467	5.7040	0.7232
1	0.2	∞	1.0755	5.7121	0.7234
1	0.1	∞	1.1110	5.7221	0.7235
1	0.0	∞	1.1525	5.7333	0.7237
1	-0.1	∞	1.1990	5.7445	0.7239
1	-0.2	∞	1.2499	5.7545	0.7240
1	-0.3	∞	1.3045	5.7624	0.7241
5	0.3	8.9935	0.5623	0.0000	0.5890
5	0.2	8.3279	0.6088	0.0000	0.6384
5	0.1	7.7058	0.6528	0.0000	0.6860
5	0.0	7.1189	0.6946	0.0000	0.7323
5	-0.1	6.5645	0.7342	5.5912	0.7515
5	-0.2	6.0437	0.7714	6.0890	0.7603
5	-0.3	5.5603	0.8060	6.4745	0.7658
10	0.3	7.2354	0.6149	0.0000	0.7227
10	0.2	6.9575	0.6458	0.0000	0.7527
10	0.1	6.6704	0.6745	0.0000	0.7823
10	0.0	6.3764	0.7013	5.2860	0.8078
10	-0.1	6.0794	0.7261	6.1940	0.8180
10	-0.2	5.7845	0.7490	6.9010	0.8245
10	-0.3	5.4980	0.7696	7.4664	0.8282

Table 4. Optimal opportunity-based age replacement time with Gaussian copula ($c_R = 1$, $c_T = 5$, $c_F = 10$).

		Replacement First		Replacement Last	
$MTTOR$	ρ_S	t_0^*	$C_1(t_0^*)$	t_0^*	$C_2(t_0^*)$
1	0.3	∞	1.0467	12.3076	0.9667
1	0.2	∞	1.0755	12.3077	0.9667
1	0.1	∞	1.1110	12.3079	0.9667
1	0.0	∞	1.1525	12.3081	0.9667
1	-0.1	∞	1.1990	12.3082	0.9667
1	-0.2	∞	1.2499	12.3083	0.9667
1	-0.3	∞	1.3045	12.3083	0.9667
5	0.3	20.9367	0.5888	0.0000	0.5890
5	0.2	20.5813	0.6382	0.0000	0.6384
5	0.1	20.1118	0.6859	0.0000	0.6860
5	0.0	19.5092	0.7323	0.0000	0.7323
5	-0.1	18.7582	0.7776	0.0000	0.7777
5	-0.2	17.8467	0.8223	0.0000	0.8223
5	-0.3	16.7665	0.8664	0.0000	0.8665
10	0.3	15.7362	0.7170	0.0000	0.7227
10	0.2	15.7377	0.7483	0.0000	0.7527
10	0.1	15.6258	0.7788	0.0000	0.7823
10	0.0	15.3918	0.8089	0.0000	0.8117
10	-0.1	15.0275	0.8388	0.0000	0.8412
10	-0.2	14.5265	0.8686	0.0000	0.8708
10	-0.3	13.8849	0.8985	0.0000	0.9006

Tables 3 and 4 show the optimal opportunity-based age replacement time t_0^* and the minimum expected cost under the opportunity-based age replacement first and last policies with $c_T = 2$ and $c_T = 5$, when the Gaussian bivariate copula is applied. Compared to the results with FGM bivariate copula, almost all the values are similar to those in Tables 1 and 2. Thus the tendency of optimal policies is almost similar to the case of FGM copula. In other words, the optimal policy is strongly dominated by the correlation between the failure time and the opportunity arrival time, but does not depend on the dependency mechanism between them.

5. Concluding Remarks

In this paper we have extended the existing opportunity-based age replacement policies by taking account of dependency between the failure time and the arrival time of a replacement opportunity for one unit system. We have characterized two cost-optimal age replacement policies which minimize the relevant expected costs, with the hazard gradient, which is a vector-valued bivariate hazard rate. In numerical examples with the FGM copula and the Gaussian copula, we have investigated the dependence of correlation between the failure time distribution and the arrival time of replacement opportunity on the optimal replacement policies. As a result, the correlation strongly affects the optimal opportunity-based age replacement policies. In particular, when the correlation is negative, the optimal policy should be carefully determined compared to the case where the correlation is positive.

Although we have given only two representative examples on the FGM bivariate copula and the Gaussian bivariate copula in this paper, the other bivariate distributions can be also considered in the similar framework. Especially, when the bivariate distribution is discontinuous such as the Marshall-Olkin family, the relevant expected cost has to be modified at $X = Y$. In addition, in our cost assumption on $c_T > c_R$, there exists the case where another opportunity-based age replacement policy by Dekker and Dijkstra[4] can become optimal. Such comprehensive researches should be reported in the future.

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